Mathematical League of University of Lodz

Series III 24/25

For every exercise you can get max. 10. p. Solutions should be delivered on paper (every task on the separate piece of paper) to the room B207 or electronically on the address: piotr.nowakowski@wmii.uni.lodz.pl. Deadline: 28.02.25.

Exercise 1. *Find all natural numbers n satisfying the following implication for all $a, b \in \mathbb{N}$:*

$$11|(a^n + b^n) \Rightarrow (11|a \wedge 11|b)$$

Exercise 2. *In certain land there is some finite number of hunters. Every set of hunters is regarded as effective or ineffective. We know that*

a) the set of all hunters is effective;

b) if the sets of hunters A and B are ineffective, then the set $A \cup B$ is also ineffective;

c) the set of hunters A is effective if and only if the set A' of remaining hunters is ineffective.

Prove that there is exactly one hunter s which is effective (that is, the set $\{s\}$ is effective).

Exercise 3. *Denote by $B(x, r)$ the open ball in $\mathbb{R}^2$ with the center in x and radius r. Let $A \subset \mathbb{R}^2$, $k \in \mathbb{N}$, $x_1, x_2, \ldots, x_k \in \mathbb{R}^2$, $r_1, r_2, \ldots, r_k > 0$ be such that*

$$A \subset \bigcup_{i=1}^{k} B(x_i, r_i).$$

Show that there is $I \subset \{1, 2, \ldots, k\}$ such that the balls $B(x_i, r_i)$ for $i \in I$ are pairwise disjoint and

$$A \subset \bigcup_{i \in I} B(x_i, 3r_i).$$